Mastering Exponential & Logarithmic Equations

Caveat

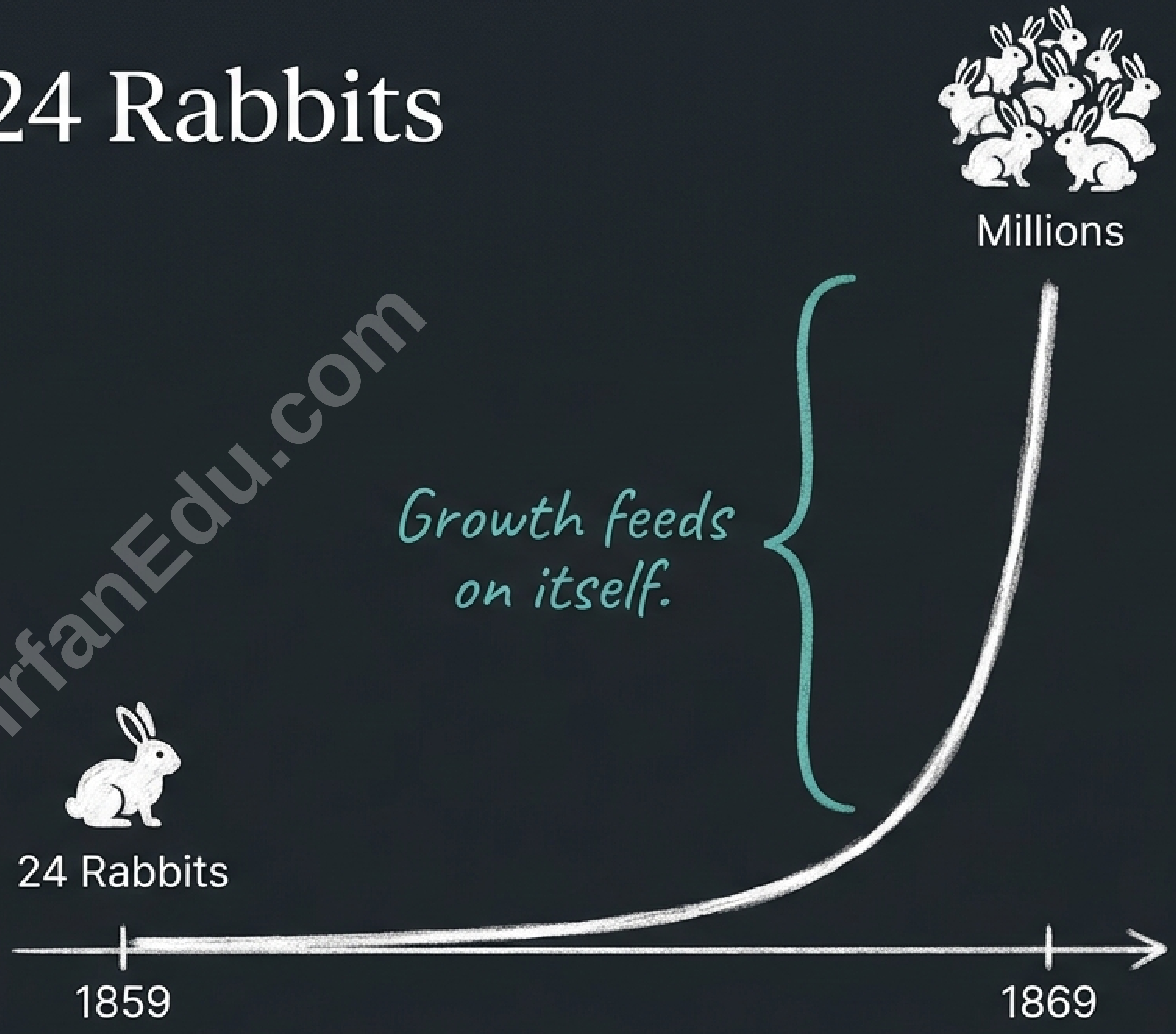
A Step-by-Step Guide to Algebra's Most Powerful Growth Models

From rabbit populations to interest rates, this is how the world grows.

It Started with 24 Rabbits

In 1859, an Australian landowner released 24 rabbits for hunting. With abundant food and few predators, the population didn't just add up; it exploded.

This is the power of the Exponential Function.



The Variable Has Moved Upstairs

Base (The Foundation) → $b^{x} = y$ ← Exponent (The Unknown)

$x^2 = 4$	$2^x = 4$
Algebraic (Variable is Base)	Exponential (Variable is Power)

Standard algebraic rules don't apply here yet. We need new tools.

Strategy 1: The One-to-One Property

If the foundations (bases) are identical, the structures (exponents) must be equal.

If $b^S = b^T$, then $S = T$

$$2^{x-1} = 2^{2x-4}$$

$$\cancel{2}^{x-1} = \cancel{2}^{2x-4}$$

$$x - 1 = 2x - 4$$

$$x = 3$$

Check:

$2^{3-1} = 2^2 = 4.$ Correct.

Strategy 2: The Disguised Base

Rewriting to find a common parent.

(2^3) (2^4) *Common Parent: 2*

$$8^{x+2} = 16^{x+1} \longrightarrow (2^3)^{x+2} = (2^4)^{x+1} \longrightarrow 2^{3x+6} = 2^{4x+4} \longrightarrow$$

Power Rule (Multiply Exponents)

$$3x + 6 = 4x + 4$$
$$\Rightarrow 6 - 4 = 4x - 3x$$
$$2 = x$$
$$x = 2$$

Check:

$$8^{2+2} = 8^4 = (2^3)^4 = 2^{12}$$
$$16^{2+1} = 16^3 = (2^4)^3 = 2^{12}$$

Correct.

When the Bases Won't Cooperate

The Dead End

$$3^{x+1} = -2$$

No Solution. Exponential functions only output positive values. Graphs never touch negative Y.

The Conflict

$$5^{x+2} = 4^{x}$$

5 and 4 share no common factors. We cannot match bases.

We need a universal key: The Logarithm.

Strategy 3: The Logarithm Key

Moving the variable from the attic to the ground floor.

$$\ln(5^{x+2}) = \ln(4^{x}) \quad \ln(5) = \ln(4)$$

$$(x+2)\ln(5) = x \cdot \ln(4)$$

Expand and Solve: $x \cdot \ln(5) + 2\ln(5) = x \cdot \ln(4)$

The Natural Language of Growth (Base e)

Inter

Euler's number ($e \approx 2.718$) tracks continuous growth. Its inverse is the Natural Log (ln).

$$100 = 20e^{2t}$$

$$5 = e^{2t}$$

← Isolate first! Divide by 20

$$\ln(5) = \ln(e^{2t})$$

← Apply natural log

$$\ln(5) = 2t$$

$$t = \frac{\ln(5)}{2}$$

Flipping the Script: Solving Log Equations

The Dictionary

$$\log_b(x) = y \Leftrightarrow b^y = x$$

$$2\ln(x) + 3 = 7$$

$$2\ln(x) = 4 \rightarrow \ln(x) = 2$$

Always isolate the log expression fully before converting!

$$e^2 = x$$

Logarithmic One-to-One Property

$$\ln(x^2) = \ln(2x + 3)$$

Caveat: Both sides are natural logs.

$$\ln x^2 = 2x + 3\ln$$

Quadratic: $x^2 - 2x - 3 = 0$

Factors: $(x - 3)(x + 1) = 0$

Solutions: $x = 3$ and $x = -1$

Always check for extraneous
solutions in the origina

The Danger Zone: Extraneous Solutions

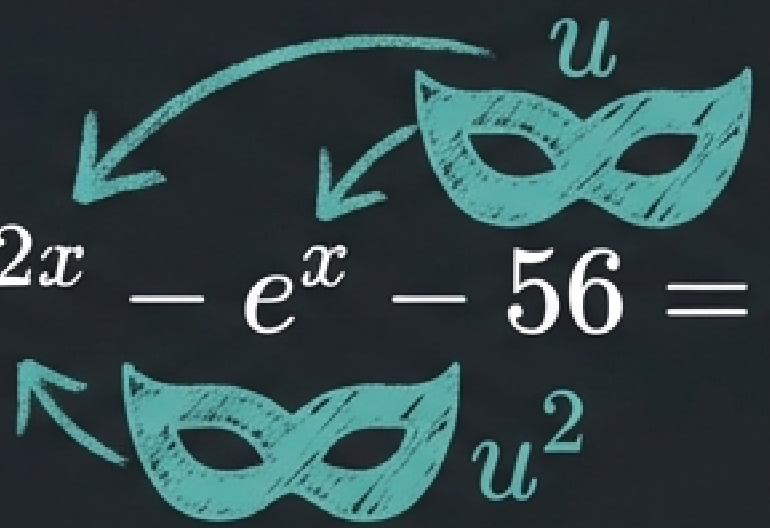
Logs cannot accept negative arguments.

Testing the Solutions	Testing the Solutions
Checking x=3	Checking x=-1
$\ln(3^2) = \ln(2(3)+3)$	$\ln((-1)^2) = \ln(2(-1)+3)$
$\ln(9) = \ln(9)$	$\ln(1) = \ln(1)$
VALID	VALID

Caveat: *Wait! x was negative (-1), but the argument became positive (1). That is allowed. We only reject if the final argument is ≤ 0.*

Advanced Technique: The Quadratic Disguise

Problem: $e^{2x} - e^{x} - 56 = 0$



$$e^{2x} - e^{x} - 56 = 0$$

New Equation: $u^2 - u - 56 = 0$

$$(u-8)(u+7) = 0 \rightarrow u = 8, u = -7$$

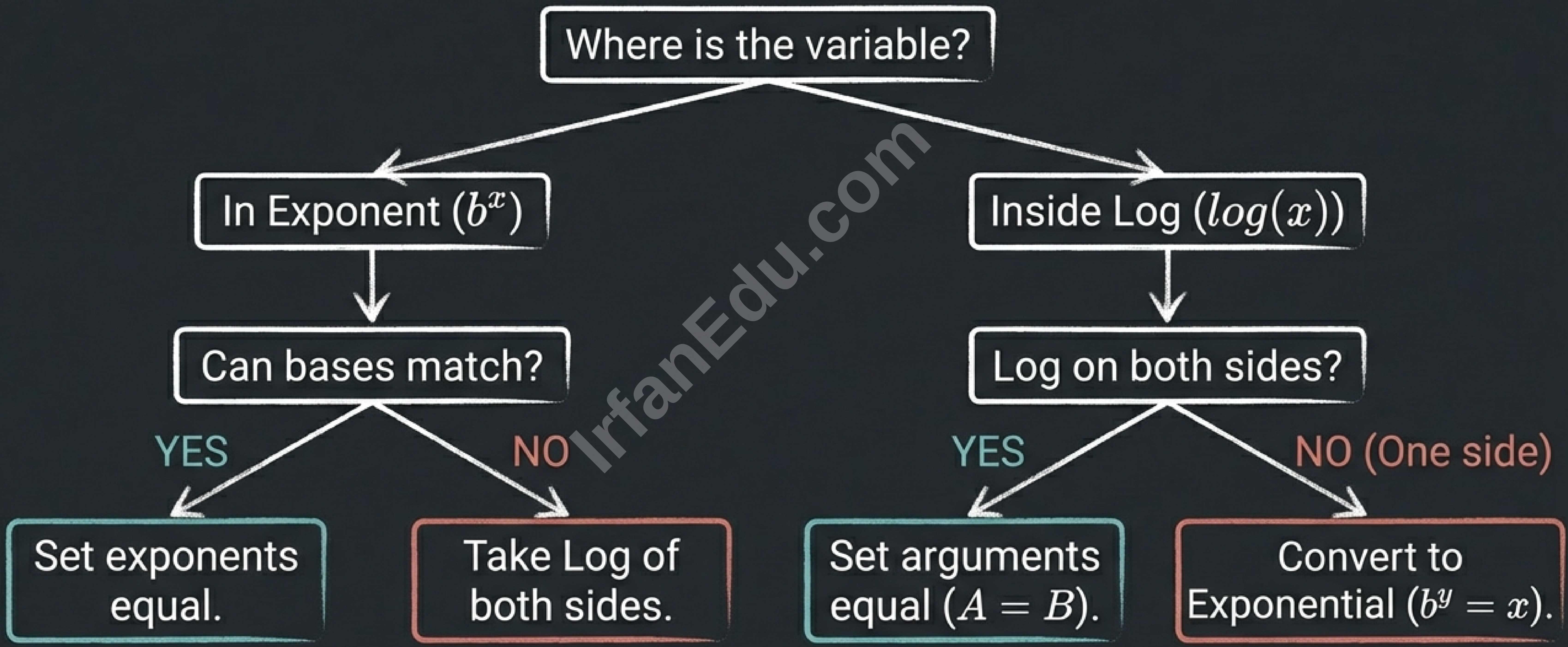
Unmasking:

$e^{x} = 8 \rightarrow x = ln(8)$ Valid

$e^{x} = -7$

Impossible. Exponentials must be > 0.

The Solver's Decision Tree



Regardless of path: ALWAYS VERIFY.

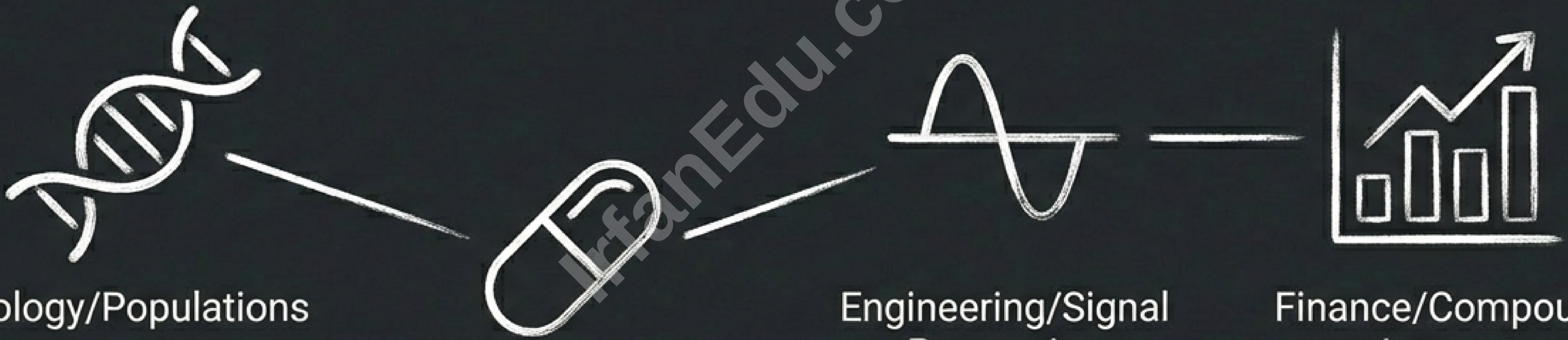
Pitfall Checklist

- [ ] **The Negative Argument:** Did you plug the answer back in? If a log argument is ≤ 0, reject it.
- [ ] **The Positive Output:** Remember that b^x can never equal a negative number.
- [ ] **The Isolate Step:** Is the log completely isolated before you convert it?
- [ ] **The Power Rule:** Did you use parenthesis? $(x+2)\ln(5)$, not $x+2\ln(5)$.

Always check your work!

Modeling the Universe

Mastery of these mechanics is the gateway to understanding how the world changes.



Understand the method. Practice the mechanics. Verify the result.